determined for analytic surfaces and reasonably accurate results (maximum error of less than 20 percent) for ratios of the order of twice this limit. This conclusion is significant because some of the previous investigators [7], [10] have suggested that the method is not applicable to such nonanalytic surfaces.

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Communications

Calculation of the Radiation Resistance of Loop Antennas with Sinusoidal Current Distribution

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Abstract—**Loop antennas with dimensions comparable to the wavelength are sometimes of practical interest. In order to use such antennas, design engineers would need simple formulas or data about the radiation resistance of such antennas. For the reason, that nothing else has been known, the radiation resistance has been calculated by means of a digital computer.**

I. INTRODUCTION

In normal radio communication, loop antennas are seldom used, and in most of these cases they serve only for receiving cases. The reasons are that mostly loop antennas have very little efficiency, especially when they are driven with uniform current (i.e., the circumference of the loop is small compared with the wavelength). It would be better, especially for all transmitting cases, if the circumference of the loop were comparable with the wavelength. The radiation resistance of the loop would be much greater than the ohmic losses and, therefore, the efficiency would increase. Unfortunately a solution of the radiation resistance of loop antennas with nonuniform current is not known. Therefore the radiation resistance of loop antennas with sinusoidal current distribution has been calculated (sinusoidal current distribution is a reasonable assumption for loops with a circumference up to about one wavelength).

II. RADIATION RESISTANCE OF LOOP ANTENNAS WITH UNIFORM CURRENT

The radiation resistance of loop antennas with uniform current and dimensions that are small compared to the wavelength can easily be calculated and is well known [1]. The solution for this case is

$$R_s = 197 \left(\frac{l}{\lambda}\right)^4 \Omega \qquad (1)$$

where

R_s radiation resistance
l circumference of the loop
λ wavelength.

Equation (1) is valid for all cases where $l/\lambda < 1/4$, i.e., where the current distribution is not differing essentially from the constant case.

In [2] the radiation resistance of loop antennas with uniform current and, compared to the wavelength, great dimensions, has been calculated. But this does not solve the problem; because in real physics, there will necessarily result an—theoretically—ideal sinusoidal current distribution. Sometimes there is proposed, to enforce a uniform current distribution by means of multiple feeding. In most cases this is not practicable since there would be paralleled much little antenna sections. Each of these sections has by reason of the dividing, a radiation resistance, which is much smaller than the radiation resistance of the whole antenna. Therefore the terminal resistance will become very small, and it will be difficult to match the antenna. Moreover the efficiency will decrease by reasons of the long and numerous section feeders. In Fig. 1 there is shown the calculated radiation resistance after [2] in the case of a uniform current distribution.

III. SINUSOIDAL CURRENT DISTRIBUTION

The radiation resistance R_s of a thin ideal conducting loop antenna can be determined with the radiated power N and the current amplitude I_0 by

$$N = \tfrac{1}{2} I_0^2 R_s. \qquad (2)$$

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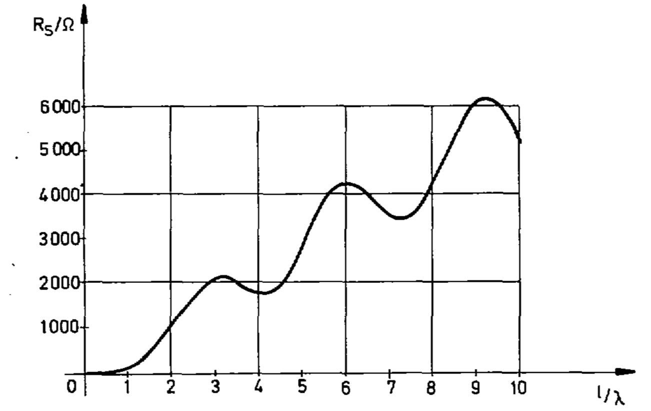

Fig. 1. Radiation resistance of loop antenna with uniform current distribution.

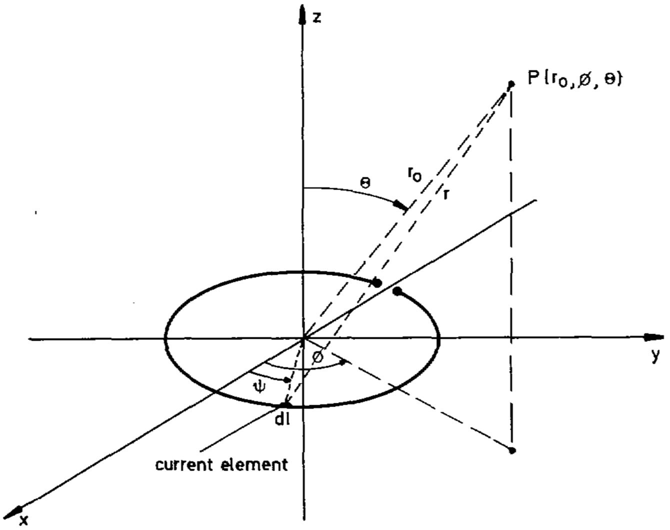

Fig. 2. Orientation of loop antenna.

The true power density P, which is radiated in radial direction, can be determined with Poynting's vector by

$$P = \tfrac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^* = \frac{1}{2Z_0}\boldsymbol{E} \times \boldsymbol{E}^* \tag{3}$$

where

$\boldsymbol{E}$ electric field intensity
$\boldsymbol{H}$ magnetic field intensity
$\boldsymbol{H}^*$ conjugate complex value of $\boldsymbol{H}$
Z_0 $120\pi\Omega$, characteristic impedance of the free space.

For a more simple calculation it is advantageous to substitute the loop antenna by a chain of many little elementary dipoles which have all constant, but differing from the neighbor current distribution. By summing up the field intensities of these elementary dipoles one gets the field distribution of the whole loop antenna. Integrating Poynting's vector over a spherical surface will deliver the radiated power.

The analysis procedure for getting the final integral whose solution delivers the radiation resistance is well known and often described in antenna literature. Therefore, the procedure of getting the integral is only shortly referred. The loop antenna shall be orientated as it is shown in Fig. 2.

The current distribution is described by

$$\begin{aligned} I(\psi) &= I_0 \cos\left(\psi \frac{l}{\lambda}\right) \\ &= I_0 \cos\left(2\pi \frac{a}{\lambda} \psi\right) \end{aligned} \tag{4}$$

where a is the radius of the loop and l is the circumference of the loop. (For loops with dimensions much greater than one wavelength this assumption is quite theoretical, nevertheless the results were calculated till $l/\lambda = 10$ and are plotted till $l/\lambda = 2.5$. This may better help to show the differences between the two modes).

The loop is divided in many elementary dipoles with the length dl, and the resulting field components $\boldsymbol{E}_\theta$ and $\boldsymbol{E}_\phi$ are determined. One elementary dipole is marked in Fig. 2.

Considering the variation of the angle ψ and the varying distances r of the current elements from the regarded coordinates of the point $P\,(r_0, \phi, \theta)$, one gets for the far-field from one element

$$\begin{aligned} d\boldsymbol{E}_\theta = -jZ_0 \frac{I_0 a \cos(l\psi/\lambda)}{2\lambda} \sin(\phi - \psi) \cos\theta \\ \cdot \frac{e^{-jkr_0}}{r_0} e^{jka \sin\theta \cos(\phi - \psi)}\, d\psi \end{aligned} \tag{5}$$

$$\begin{aligned} d\boldsymbol{E}_\phi = -jZ_0 \frac{I_0 a \cos(l\psi/\lambda)}{2\lambda} \cos(\phi - \psi) \\ \cdot \frac{e^{-jkr_0}}{r_0} e^{jka \sin\theta \cos(\phi - \psi)}\, d\psi. \end{aligned} \tag{6}$$

To get the complete field, it is necessary to integrate all particular field-components.

The power density P is delivered by

$$P = \frac{1}{2}\frac{1}{Z_0}\boldsymbol{E}_\theta \boldsymbol{E}_\theta^* + \frac{1}{2}\frac{1}{Z_0}\boldsymbol{E}_\phi \boldsymbol{E}_\phi^*. \tag{7}$$

With (7) one can destinate whole the power N which is radiated

$$\begin{aligned} N &= \oint P\, dA \\ &= \frac{2}{Z_0}\int_{\phi=0}^{\pi}\int_{\theta=0}^{\pi/2} \{\boldsymbol{E}_\theta \boldsymbol{E}_\theta^* + \boldsymbol{E}_\phi \boldsymbol{E}_\phi^*\} r^2 \sin\theta\, d\theta\, d\phi. \end{aligned} \tag{8}$$

With (2)–(7) one gets, after some conversions,

$$\begin{aligned} R_s = Z_0 \left(\frac{a}{\lambda}\right)^2 \int_{\phi=0}^{\pi}\int_{\theta=0}^{\pi/2} (J_1^2 + J_2^2 + J_3^2 + J_4^2) \\ \cdot \sin\theta\, d\theta\, d\phi \end{aligned} \tag{9}$$

wherein J_i are integrals of the form

$$\begin{aligned} J_1 &= \int_{-\pi}^{\pi} \cos(ka\psi) \sin(\phi - \psi) \cos\theta \\ &\qquad \cdot \cos(ka \sin\theta \cos(\phi - \psi))\, d\psi \\ J_2 &= \int_{-\pi}^{\pi} \cos(ka\psi) \sin(\phi - \psi) \cos\theta \\ &\qquad \cdot \sin(ka \sin\theta \cos(\phi - \psi))\, d\psi \\ J_3 &= \int_{-\pi}^{\pi} \cos(ka\psi) \cos(\phi - \psi) \\ &\qquad \cdot \cos(ka \sin\theta \cos(\phi - \psi))\, d\psi \\ J_4 &= \int_{-\pi}^{\pi} \cos(ka\psi) \cos(\phi - \psi) \\ &\qquad \cdot \sin(ka \sin\theta \cos(\phi - \psi))\, d\psi. \end{aligned} \tag{10}$$

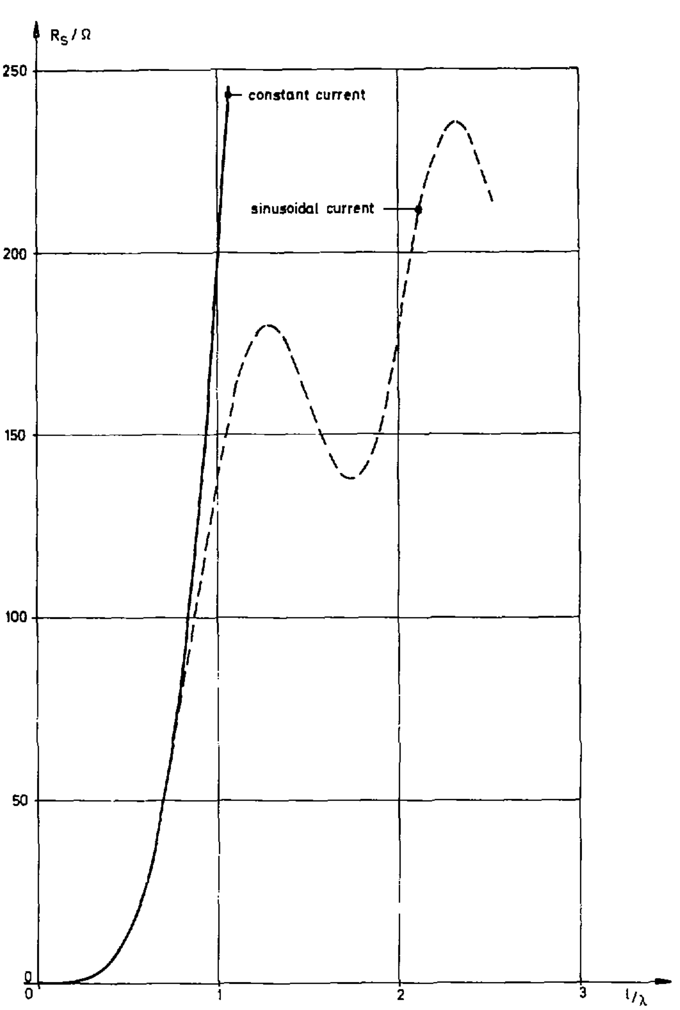

Fig. 3. Radiation resistance of loop antenna with sinusoidal current distribution versus constant current distribution.

The solution of this triple integral is leading already after the first integral to Bessel functions of the order $\mathcal{J}_{a+jb}(x)$, where a and b may not be integer. A retrospective view at the way of solution in [2] shows, that the author has calculated just the last problem, which can be solved closed.

In [2] it has been possible, to convert the Bessel functions (for constant current their order is integer) into sums, and therefore they remain integrable, not so in this special case. For sinusoidal current we have solved the integral (9) with a digital computer.

IV. Calculation of the Radiation Resistance Using a Computer

By solving the problem with a digital computer special attention had to be paid in order to get sufficient accuracy. Normally the accuracy of the result will increase with an increasing number of probes. On the other hand, the accuracy will decrease if there are too many probes because of the accumulation of rounding errors and other inaccuracies of the computer. An optimal number of probes had to be found. It has been shown, that the accuracy is strongly dependent on the number of probes in ψ. Less critical are the other dimensions ϕ and θ. Moreover it has to be observed, that there will be no interfering between probe number and wave number which would cause incorrect extinction of some field components. Therefore the probe distances have been chosen not constant, but increasing with a special function. The resulting faults can be eliminated by applying a corregating weighting function.

In Fig. 1 the calculated radiation resistance is shown versus the wave number around the loop (dotted line). If one compares this with the other curve, it can be seen, that the radiation resistance for sinusoidal current distribution is expectively much less. In the region where it is $l/\lambda \ll 1$, both curves are approximately equal. There also appears an oscillation around a medium value (for $l/\lambda \gg 1$ compare in [2]), but now this oscillation has a period of 1 in contrast to the period π. This leads to the explanation that in the case of a loop antenna with uniform current distribution the oscillation is the effect of adding or subtracting field components, which are excited from diametric antenna segments. In the case of sinusoidal current distribution the oscillating is caused by adding or subtracting field components, which now result from neighboring segments. This effect is well known from linear antennas.

V. Conclusion

It has been shown, that the calculation of the radiation resistance of a loop antenna with sinusoidal current distribution cannot be solved closed. Therefore the solution of the problem has been carried out by the aid of a digital computer.

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Measured Field of a Directional Antenna Submerged in a Lake

L. C. SHEN, R. W. P. KING, and R. M. SORBELLO

Abstract—**Field patterns of a VHF directional antenna submerged in a lake are shown with the receiver both under and over the surface of the water. The measured relative strength of the electric field as a function of the radial distance from the source is also shown. The rate of decrease with distance of the radial component of the field in the water agrees with theoretical predictions.**

I. Introduction

The transmission of electromagnetic waves over the surface of the earth has been a subject of scientific interest and technical importance since the formulation of the basic theory by Sommerfeld in 1909. At that time and for many years the primary motivation was radio communication between transmitting and receiving antennas erected above the earth. The application of the theory to subsurface communication between two submarines began with a thesis by Moore in 1951 and two unpublished reports by Baños and Wesley in 1953 and 1954. Expanded and corrected treatments are in the more recent literature [1]–[3]. In particular, the comprehensive work of

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